Throughput Analysis of Proportional Fair Scheduling For Sparse and Ultra-Dense Interference-Limited OFDMA/LTE Networks

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Abstract

Various system tasks like interference coordination, handover decisions, admission control etc. in current cellular networks require precise mid-term (spanning over a few seconds) performance models. Due to channel-dependent scheduling at the base station, these performance models are not simple to obtain. Furthermore, LTE cellular systems are interference-limited, hence, the way interference is modelled is crucial for the accuracy. In this paper we present a closed-form analytical model for the throughput expectation of proportional fair scheduling in OFDMA/LTE networks. The model takes into account a precise SINR distribution as well as it considers limitations with respect to modulation and coding, as applied in LTE networks. Furthermore, the analysis is extended also for ultra-dense deployments likely to happen in the 5-th generation of cellular networks. The resulting analytical performance model is validated by means of simulations considering realistic network deployments. Compared with related work, the model introduced in this paper demonstrates a significantly higher accuracy for long-term throughput estimation.

I. Introduction

Proportional fair scheduling (PFS) has found a wide applicability in several generations of wireless cellular networks. Originally, it was proposed in [1] and then further adapted for CDMA and OFDMA-based systems respectively in [2], [3]. It is widely used due to its ability to schedule mobile terminals in peak channel state realizations while providing a balanced throughput distribution among a set of terminals with strongly different average channel qualities. This has lead to the use of PFS also as a reference scheduler in many other investigations. Therefore, exact closed-form analytical models of the terminal throughput under PFS would be of significant interest.

However, PFS throughput analysis turns out to be a challenging task. This is mainly due to the fact that decisions of dynamic schedulers like PFS are based on instantaneous channel states, which are of random nature. Specifically, under PFS the instantaneous channel states (either characterized by the rate or by the signal-to-interference-and-noise ratio) are normalized by their time-averages over a certain window size. The scheduler then assigns the transmission resources which have the highest values according to this normalization. As a consequence, "above-average" channel states are likely to be scheduled and thus, the statistics of the scheduled resources change in a beneficial way. To determine the average rate obtained under PFS, one needs to compute an integration over these modified statistics of the channel states. This integration is typically complex due to three reasons: (I) The presence of interference makes the analysis of unscheduled channel statistics difficult to handle; (II) The statistics of the assigned resources can not be integrated; (III) System features like delayed feedback or common modulation and coding for a set of co-scheduled resource blocks further complicate the matter.

The first aspect is most challenging for interference-limited scenarios where the random interactions of the interfering signals with the signal of interest needs to be modeled. Furthermore, when rate is used as scheduling metric as is the case for rate-based PFS, one further needs to take into account the SINR-to-rate mapping of the corresponding system. Confronted with these issues, related work on PFS analysis mostly rely on approximation methods for describing the PDF and CDF of the *scheduled* channel states either given by SINR or rate. For example, [4], [5] uses Gaussian approximation to describe the instantaneous rate realizations. Meanwhile, in other works [6], [7], [8], [9] total interference, although time varying, is simply considered as an additional constant noise or not considered at all in [10], [11], [7]. Therefore, the instantaneous SINR is modeled as an exponentially distributed random variable. While these approaches help in gathering first insights on PFS, in previous work [12], [13] we have shown that for interference-limited scenarios such approximations can lead to severe expected rate errors of 50 % and more. Another work considering the stochastic variation of interfering power is presented in [14]. Nevertheless, for the computation of a closed-form throughput model of PFS, that the long-term scheduling priorities are independently and identically distributed (i.i.d.). While this assumption holds for homogeneous propagation scenarios with a relatively small deviation of shadow fading, its accuracy degrades considerably for realistic urban scenarios characterized from very inhomogeneous propagation conditions. Especially, for such scenarios the model accuracy is crucial for the performance of long term radio resource management such as: semi-static inter-cell interference coordination, admission control, etc. Hence, better methods for determining the rate expectations are required. This is further stressed by the fact that practical systems, such as LTE, have constraints on the modulation and coding scheme (MCS) selection for the scheduled resources. For a set of co-scheduled resources, the transmitted transport block is allowed to use only one MCS for all scheduled resources. For a robust transmission of the transport block one needs to consider the joint realizations of the individual subcarriers. Consequently, this influences the performance of PFS as the *joint* statistics of different co-scheduled resources need to be included accurately in the analysis as well. Hence, simplifications on the channel quality statistics can lead to considerable performance degradation. For example, [9] considers the common MCS constraint, but it ignores the interference variation over time by considering it as constant. We believe that there is still room for improvement on the accuracy of PFS modeling in interference limited scenarios and this is to be achieved by considering an accurate distribution of the channel quality statistics. To the best of our knowledge, no previous work has considered this aspect so far.

In this paper we address PFS analysis for interference limited scenarios. We consider an accurate distribution of the SINR and compute its PDF and CDF transformation due to the dynamic selection of PFS. Based on these functions we introduce closed-form analytical models for the expected throughput in OFDMA systems operating by PFS. The introduced analytical expressions take into account LTE protocol constraints on modulation and coding schemes. We also perform an asymptotic analysis, letting the number of interferers grow to infinity and considering the respective expected throughput of terminals exposed to the joint interference process. After presenting the mathematical derivations, we evaluate them for typical deployments of OFDMA networks. For time- and frequency-correlated propagation scenarios, operating either under channel-based or rate-based PFS, our numerical evaluations show that the expressions introduced here provide the highest accuracy compared with the common approximation methods from related work. The observed performance differences are quite significant, especially with respect to the distribution of the relative error, implying that schemes such as admission control, interference coordination or load balancing easily over- or underestimate the system performance if they are based on the state-of-the-art models. Finally, our presented contribution builds on top of own previously published work [15], [12]. However, in this paper we provide in-depth mathematical derivations leading to closed-form expressions for the rate expectation of PFS either in an ideal scenario as well as a more accurate model for LTE. In addition, we provide an asymptotic analysis for ultra-dense deployments such that - overall - this paper significantly extends and concludes our previous contributions in [15], [12].

Our remaining work is structured in the following way. In Section II we introduce the system model. Then, in Section III we present the problem statement and discuss the major approaches from related work. Section IV presents then our main analytical contributions as well as the asymptotic analysis. We then validate and benchmark our analytical expressions in comparison to state-of-the-art in Section V. Finally, conclusions are drawn in Section VI

II. System Model

We consider the downlink communication of a multi-cellular LTE-like deployment operating on single-input-single-output (SISO) links and on orthogonal frequency division multiple access (OFDMA). Our focus is on the performance of a set $\mathcal{J}$ of terminals which are associated to a designated base station (with index 0). The downlink communication of the considered base station to the set of terminals is subject to interference from I neighboring base stations. Time is slotted into so called transmission time intervals (TTI) with duration T_{TTI} $[\text{ms}]$ and index t. The system utilizes a bandwidth of B $[\text{Hz}]$ which is split into N chunks of subsequent sub-carriers also known as resource blocks. More precisely, a resource block (RB) n is formed out of N_{S} consecutive sub-carriers in the frequency domain and N_{C} symbols in the time domain. Every time slot t the base station, being the central coordination point of transmissions in the cell, dynamically allocates these resource blocks to mobile terminals in the cell.

Every TTI the base station performs resource allocation according to a channel-state dependent proportional fair scheduling (PFS) algorithm. Assuming instantaneous and unquantized channel state information at the base station and full-buffer traffic model, the scheduling decisions are then exclusively dependent on the instantaneous channel states as well as on the time-average of these states. The channel state is defined in the following by the signal-to-interference-and-noise ratio (SINR) $Z_{j,n}(t)$ for terminal j with respect to RB n. In the frequency domain, the SINR is considered as constant within the bandwidth and time duration of a resource block, whereas its realization in different time slots is modeled as a random variable and given by:

$$Z_{j,n}(t) = \frac{p_{0,j,n} X_{0,j,n}(t)}{\sum_{i=1}^{I} p_{i,j,n} X_{i,j,n}(t) + N_{\text{o}}} \; . \tag{1}$$

Here, N_{o} denotes the noise power, while, $p_{0,j,n}$ and $p_{i,j,n}$ respectively represent the average received power of terminal j from the serving base station ($i = 0$) and interfering base stations ($i > 0$) regarding resource block n. Furthermore, $X_{i,j,n}(t)$ denotes the fading component which is random and varies from time slot to time slot and also from resource block to resource block. We adopt the well known Rayleigh-fading model for these random variables, leading to an exponential distribution of $X_{i,j,n}(t)$ with unit mean. For validation purposes we also consider correlation in time and frequency of $X_{i,j,n}(t)$, but do not characterize this further in the analytical models. Due to the randomness of the fading gains $X_{i,j,n}(t)$, $Z_{j,n}(t)$ has a specific distribution which we discuss in Section IV.

The average received power $p_{i,j,n}$ depends in general on the path-loss $\bar{h}_{i,j}^{\text{PL}}$ and shadowing $\bar{h}_{i,j}^{\text{SH}}$ regarding the corresponding base station as well as on the transmission power per resource block $P_{n,i}$, i.e.,: $p_{i,j,n} = \bar{h}_{i,j}^{\text{SH}} \cdot \bar{h}_{i,j}^{\text{PL}} \cdot P_{n,i}$. In this work we do not consider the variation of the shadow fading over the cell area. Instead, we are concerned with the rate expectation for a given shadowing and pathloss realization. In the following we assume the average received power to remain constant for the time span of interest to our analysis in Section IV. In practice this is realistic for a few hundred TTIs and more, depending on the mobility of the assumed scenario.

Coming back to the operation of PFS, given the SINR realizations $Z_{j,n}(t)$, the channel-state dependent PFS algorithm operates as follows. Let:

$$\bar{Z}_{j,n}(t) = \frac{1}{W} \sum_{i=t-W}^{t-1} Z_{j,n}(i) \tag{2}$$

denote the average SINR during the last W time slots on resource block n for terminal j. Then $\widehat{Z}_{j,n}(t)$ denotes the scaled SINR defined as:

$$\hat{Z}_{j,n}(t) = \frac{Z_{j,n}(t)}{\bar{Z}_{j,n}(t)}. \tag{3}$$

Based on this, the PFS algorithm decides on scheduling resource block n to terminal j if it has the best scaled SINR among the other terminals in its cell. We denote this scheduling decision by the indicator variable $S_{j,n}(t)$ defined as:

$$S_{j,n}(t) = \begin{cases} 1, & \hat{Z}_{j,n}(t) \geq \max_{\forall g \in \mathcal{J} \setminus \{j\}} \{\hat{Z}_{g,n}(t)\} \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Once PFS is performed the amount of payload bits for transmission has to be determined. We assume the system to feature adaptive modulation and coding with M total modulation and coding schemes (MCS) available. Hence, the SINR range is split up accordingly into M different intervals $A_m = [z_m,\ z_{m+1}[$. If the SINR realization $Z_{j,n}(t)$ is within range A_m then the spectral efficiency c_m is employed according to the selected MCS. Examples of range settings z_m and corresponding spectral efficiencies c_m for LTE systems can be found in [16], [17]. We denote this system feature by a SINR-to-payload size mapping function $C(z) = N_{\text{S}} \cdot N_{\text{C}} \cdot c_m \cdot \mathbb{1}_{A_m}(z)$, where

$$\mathbb{1}_{A_m}(z) = \begin{cases} 1, & z_m \leq z < z_{m+1} \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Note that c_m is increasing in m.

If during the same TTI a set of RBs is scheduled to a specific terminal, then in LTE the *same* MCS must be used for the *whole set* of resource blocks during the downlink transmission. In our practical model we take this into account by adopting the method presented in [3] where for the set of co-scheduled RBs, the MCS is set according to the minimum value. As a result, the scheduled payload data to be transmitted during the upcoming TTI t for terminal j is given by:

$$R_j(t) = \sum_{n=1}^{N} \frac{1}{T_{\text{TTI}}} S_{j,n}(t) \min_{n} \{C(Z_{j,n}(t)) | S_{j,n}(t) = 1\}. \tag{6}$$

Other methods for MCS selection are used in [9] and [18] which are based respectively on exponential effective SINR mapping (EESM) and on mutual information effective SINR mapping (MIESM). The latter is out of the scope in our analysis. In our previous publication [19] we have shown that the minimum SINR is a good approximation of EESM method. Especially, for small sets of resource blocks this approximation is plausible as the likelihood of scheduling large sets of resource blocks is relatively small.

III. Problem Statement and Related Work

In this paper we address the expected throughput of proportional fair scheduling in interference-limited wireless channels under Rayleigh-fading of both the desired and interfering signals. Hence, we are interested in the throughput expectation over time periods for which the average received powers $p_{i,j,n}$ can be assumed to be constant. This expectation is clearly of large interest, as system tasks such as admission control, load balancing, and interference coordination rely on an accurate model of this expected throughput. Under our system model, the expected throughput for terminal j can be shown to be given by [5], [15]:

$$\mathcal{R}_j = \sum_{n=1}^{N} \frac{1}{T_{\mathrm{TTI}}} \int_0^{\infty} C(z) f_{Z_{j,n}|S_{j,n}=1}(z) \mathrm{P}(S_{j,n}=1)\,\mathrm{d}z \tag{7}$$

where $f_{Z_{j,n}|S_{j,n}=1}(z)$ is the SINR probability density function (PDF) of the scheduled resource blocks and $\mathrm{P}(S_{j,n}=1)$ is the scheduling probability of resource block n to MS j. Based on Bayes' theorem the scheduled SINR PDF is given by:

$$f_{Z_{j,n}|S_{j,n}=1}(z) = \frac{f_{S_{j,n}=1|Z_{j,n}}(z) f_{Z_{j,n}}(z)}{\mathrm{P}(S_{j,n}=1)}, \tag{8}$$

where $f_{Z_{j,n}}(z)$ is the PDF of the random SINR $Z_{j,n}$. By means of best order statistics the term $f_{S_{j,n}=1|Z_{j,n}}(z)$ is given as:

$$\begin{aligned} f_{S_{j,n}=1|Z_{j,n}}(z) &= \mathrm{P}\left(\hat{Z}_{j,n} \geq \max_{\forall g \in \mathcal{J} \setminus j} (\hat{Z}_{g,n}) | Z_{j,n} = z\right) \\ &= \mathrm{P}\left(\frac{z}{\mathrm{E}\left[Z_{j,n}\right]} \geq \max_{\forall g \in \mathcal{J} \setminus j} (\hat{Z}_{g,n})\right) \\ &= \prod_{\forall g \in \mathcal{J} \setminus j} F_{Z_{g,n}}\left(\frac{\mathrm{E}\left[Z_{g,n}\right]}{\mathrm{E}\left[Z_{j,n}\right]} z\right), \end{aligned} \tag{9}$$

where $F_{Z_{g,n}}(z)$ is the SINR CDF. Combining Equations (7),(8) and (9), the expected throughput is given as:

$$\mathcal{R}_j = \sum_{n=1}^{N} \frac{1}{T_{\mathrm{TTI}}} \int_0^{\infty} C(z) \prod_{\forall g \in \mathcal{J} \setminus j} F_{Z_{g,n}}\left(\frac{\mathrm{E}\left[Z_{g,n}\right]}{\mathrm{E}\left[Z_{j,n}\right]} z\right) f_{Z_{j,n}}(z)\,\mathrm{d}z \tag{10}$$

Clearly, (10) is not trivially solved, as in the interference case the involved PDF and CDFs are functions of random variables, leading to complex expressions over which the integral needs to be found. In the literature, this has lead - to the best of our knowledge - to several simplifications when dealing with the expectation in (10). We summarize five prominent approaches in the following.

In the first case [6], [7], [8], the random variable $Z_{j,n}$ is assumed to be exponentially distributed. The analysis in [7] concentrates on the effective SNR distribution when a single MCS scheme is used for the whole set of co-scheduled RBs. However, the analysis is performed only for noise-limited systems. The other works [6], [8] account for the interference, by simply increasing the noise power according to the average interference power. Hence, neglecting the fact that the interference power undergoes fading. More precisely, this results in the (simplified) density function:

$$f_{Z_{j,n}}(z) = \frac{1}{\tilde{Z}_{j,n}} \mathrm{e}^{-\frac{z}{\tilde{Z}_{j,n}}}\ , \tag{11}$$

where $\tilde{Z}_{j,n}$ is the SINR resulting from the average received signal strength of the signal of interest as well as the interfering signals (omitting the fast-fading component) defined as:

$$\tilde{Z}_{j,n} = \frac{p_{0,j,n}}{P_{j,n} + N_{\mathrm{o}}}, \tag{12}$$

where $P_{j,n} = \sum_{i=1}^{I} p_{i,j,n}$ is the accumulated average interfering power at the mobile terminal. While such an assumption leads to tractable analytical expressions for the integral in (10), the authors [6], [8] still propose to apply numerical methods to solve the integral. We refer to this kind of modeling as *interference as noise (IaN)* approximation.

The second approach [20], [21], [5], [22] is based on the *rate distributions* of $R_{j,n}(t)$ while borrowing some of the ideas of the IaN approximation regarding the SINR distributions of $Z_{j,n}(t)$. The main simplification comes from assuming the distribution of $R_{j,n}(t)$ to be Gaussian [23] for which the mean and variance need to be estimated. [21] proposes for this matching the following method, which is based on the IaN approximation discussed in the previous paragraph:

$$\mu_{R_{j,n}} \quad = \quad \int_0^\infty C\left(\tilde{Z}_{j,n} \cdot z\right) \cdot \mathrm{e}^{-z} \mathrm{d}z \tag{13}$$

and $\sigma^2_{R_{j,n}}$ given by:

$$\sigma^2_{R_{j,n}} \quad = \quad \int_0^\infty C\left(\tilde{Z}_{j,n} \cdot z\right)^2 \cdot \mathrm{e}^{-z} \mathrm{d}z - \mu^2_{R_{j,n}}. \tag{14}$$

Given this parameterization and using the Gaussian assumption of the rate distribution, the average rate of terminal j under PFS can then be approximated as:

$$\mathcal{R}_j \quad = \quad \sum_{n=1}^{N} \frac{1}{T_{\mathrm{TTI}}} \int_0^\infty \frac{\left(z\sigma_{R_{j,n}} + \mu_{R_{j,n}}\right)}{\sqrt{2\pi}} \mathrm{e}^{\frac{-z^2}{2}} \prod_{\forall g \in \mathcal{J} \setminus j} F_{(0,1)}\left(\frac{\mu_{R_{g,n}} \sigma_{R_{j,n}}}{\mu_{R_{j,n}} \sigma_{R_{g,n}}} z\right) \mathrm{d}z\,, \tag{15}$$

in which $F_{(0,1)}(\cdot)$ is the standard Gaussian distribution function with zero mean and unit variance. As a result, the integral in (15) ends up in a sum of Q-Functions which can be obtained by table look-ups. We refer to this approach as *Gaussian approximation*.

A third approximation is performed [14], where the long-term priority coefficients $\frac{\mathrm{E}[Z_{g,n}]}{\mathrm{E}[Z_{j,n}]}$ are approximated as independent and identically distributed (i.i.d.). Then, the (10) is brought into a form:

$$\mathcal{R}_j = \sum_{n=1}^{N} \frac{1}{T_{\mathrm{TTI}}} \int_0^\infty C(z) F_{Z_{j,n}}(z)^{|\mathcal{J}|-1} f_{Z_{j,n}}(z)\,\mathrm{d}z, \tag{16}$$

which is easier to solve than (10). We refer to this model as *i.i.d. priority approximation*.

The models introduced so far still do not consider the joint MCS constraint as in OFDMA/LTE networks. The work in [9] considers this aspect by computing the effective SINR distribution. Nevertheless, it makes the same assumption of the *interference as noise* approximation, where the time variation of the interference power is ignored. Such simplification can have a negative impact on the accuracy of the model as the MCS selection depends on the joint realization of a multiple set of subcarriers. Inaccuracies stemming from the simplification of a single subcarrier become more obvious while considering the joint statistics of sets of subcarriers. We refer to this simplification as *unique MCS and interference as noise* approximation.

Finally, a fifth approach for the expected PFS throughput is presented in [24], [25] utilizing a very simple approximation. It does not take into account neither the impact of the scheduler on the rate expectation nor the instantaneous variation of the signal of interest and interference. Instead of computing the expectation of scheduled resources, this approach simply accounts for the average SINR. For a given SINR-to-spectral efficiency mapping function $C(\cdot)$, the resulting average rates compute to

$$\mathcal{R}_j = \sum_{n=1}^{N} \frac{1}{|\mathcal{J}| \cdot T_{\mathrm{TTI}}} C\left(\tilde{Z}_{j,n}\right). \tag{17}$$

We refer to this model as *simple approximation*.

Obviously, all five major methods from literature utilize quite strong approximations in modeling the impact from interference as well as the impact from dynamic scheduling. This is clearly not desirable as it is unclear from related work how big the modeling error might become due to the different approximation models, especially in case of an interference-limited scenario. In Table I we summarize once more the state-of-the art on PFS analysis and contrast it to our contribution. In this paper we introduce a closed-form PFS model that accounts for both the MCS constraint and time-varying interfering power in OFDMA-LTE networks. Although the analysis is based on a SINR-based PFS scheduler it can be also used to approximate the expected throughput performance of rate-based PFS schedulers as performed in OFDMA-LTE networks.

Approximation:	Scheduler Impact	Interference Variation	MCS Constraint	Rate PFS	SINR PFS
Simple [24], [25]	✗	✗	✓	✗	✗
Noise Limited [10], [11], [7]	✓	✗	✓\[7]	✗	✓
Interf. as Noise (IaN) [6], [8]	✓	✗	✗	✗	✓
Gaussian [20], [21], [5], [22]	✓	✗	✗	✓	✗
I.i.d. priority approx. [14]	✓	✓	✗	✓	✗
Unique MCS & IaN [9]	✓	✗	✓	✗	✓
Our Contribution	✓	✓	✓	✓	✓

TABLE I: Comparison of related work to the contribution of this work.

IV. PFS Analysis for Interference Limited Scenarios

In this section we provide our main results. In Section IV-A, we first derive a closed-form solution of the throughput expectation considering an accurate distribution of the SINR and a relaxed constraint on MCS allocation. Then, in Section IV-B we extend the throughput expectation analysis for the given system model with the unique MCS constraint for a set of co-scheduled resource blocks. Finally, in Section IV-C we analyze the asymptotic performance of PFS for a large number of interferers, which has practical relevance for example with ultra-dense cellular deployments.

A. Throughput Expectation with Relaxed MCS Constraint

Due to a discrete number of available rates to choose, we can reformulate the rate expectation integral in (10) to:

$$\mathcal{R}_j = \frac{N_{\mathrm{S}} N_{\mathrm{C}}}{T_{\mathrm{TTI}}} \sum_{n=1}^{N} \sum_{m=2}^{M} c_m \int_{z_{m-1}}^{z_m} \prod_{\forall g \neq j \in \mathcal{J}} F_{Z_{g,n}} \left(\frac{\mathrm{E}[Z_{g,n}]}{\mathrm{E}[Z_{j,n}]} \cdot z \right) \cdot f_{Z_{j,n}}(z) \; dz \tag{18}$$

Note that in the expression above we keep the index over the resource blocks n to allow for example for different transmit power settings on different resource blocks. Now, the function under the integral sign depends only on the PDF and CDF of the SINR. Current literature on the SINR CDF [26], [14] provides expressions composed from a product of elementary functions. However, in order to solve the throughput integral (18) we need to transform the CDF expression into a sum of elementary functions. By applying partial fractional decomposition in Appendix A we obtain:

$$F_{Z_{j,n}}(z) = 1 - \sum_{i=1}^{I} U(c_{i,j,n}) \frac{1}{c_{i,j,n} + z} \mathrm{e}^{-zc_{0,j,n}}, \tag{19}$$

where $c_{i,j,n} = \frac{p_{0,j,n}}{p_{i,j,n}}$, $c_{0,j,n} = \frac{N_0}{p_{0,j,n}}$ and

$$U(c_{i,j,n}) = \begin{cases} \prod_{a=1}^{I} c_{a,j,n} \left(\sum_{g=1}^{I} \prod_{f \neq g} (c_{f,j,n} - c_{i,j,n}) \right)^{-1}, & I \geq 2 \\ c_{i,j,n}, & I = 1; \end{cases} \tag{20}$$

Taking the derivative of (19) with respect to variable z the PDF $f_{Z_j}(z)$ of the SINR is obtained as:

$$f_{Z_j}(z) = \sum_{i=1}^{I} U(c_{i,j,n}) \left(\frac{1}{(c_{i,j,n} + z)^2} + \frac{c_{0,j,n}}{c_{i,j,n} + z} \right) \mathrm{e}^{-zc_{0,j,n}}. \tag{21}$$

Although valid for an unlimited number of observations, we use the SINR expectation $\mathrm{E}[Z_{j,n}]$ to approximate the normalization factor $\bar{Z}_{j,n}$ for a limited but relatively large window size W. It follows as:

$$\mathrm{E}[Z_{j,n}] = \sum_{i=1}^{I} U(c_{i,j,n}) \mathrm{e}^{c_{i,j,n} c_{0,j,n}} \mathrm{Ei} \left(-c_{i,j,n} c_{0,j,n} \right), \tag{22}$$

where $\mathrm{Ei}(\cdot)$ is the exponential integral function.

Based on the SINR CDF and PDF, we proceed with addressing the solution of the integral in (18). For this, we decompose the product within the integral to a sum of simpler functions whose integral solution is known. Based on the steps taken in Appendix B we decompose the CDF product, multiply with the corresponding PDF and solve the elementary integrals. The indefinite integral in (18) is solved as:

$$\int \prod_{\forall g \neq j} F_{Z_g}\left(\frac{\mathrm{E}[Z_g]}{\mathrm{E}[Z_j]} \cdot z\right) f_{Z_j}(z)\,\mathrm{d}z = \tag{23}$$

$$= F_{Z_j}(z) + \sum_{k=1}^{|\mathcal{J}|-1} (-1)^k \sum_{g_1<g_2<\cdots<g_k} \sum_{r=1}^{k} \sum_{i_r=1}^{I} \sum_{l=1}^{k} \sum_{t=1}^{I} W(\hat{c}_{i_r,g_l}) U(c_{t,j}) \Bigg(-\frac{\mathrm{e}^{-Dz}}{c_{t,j}+z} +$$

$$(A-C)\mathrm{e}^{\hat{c}_{i_r,g_l}D}\mathrm{Ei}\left(-D(\hat{c}_{i_r,g_l}+z)\right) + (B+C+AD)\mathrm{e}^{c_{t,j}D}\mathrm{Ei}\left(-D(c_{t,j}+z)\right)\Bigg), \tag{24}$$

where $A = \frac{1}{\hat{c}_{i_r,g_l}-c_{t,j}}$, $B = \frac{1}{\hat{c}_{i_r,g_l}c_{t,j}} - \frac{c_{t,j}}{c_{i_r,g_l}(c_{t,j}-c_{i_r,g_l})} - \frac{1}{c_{t,j}(c_{i_r,g_l}-c_{t,j})}$, $C = \frac{c_{0,j}}{\hat{c}_{i_r,g_l}-c_{t,j}}$, and $D = \sum_{l=1}^{k} \hat{c}_{0,g_l} + c_{0,j}$. Based on this expression, we finally obtain a solution of the expression in (18) by sequentially putting in the limits for the different MCS combinations. Next, we focus on the analysis of the unique MCS constraint, which relies to some extent on the analysis presented above.

B. Throughput Expectation for the Unique MCS Constraint

Recall that in practical systems, the resource blocks are no longer assigned an MCS individually. Instead a common scheme is selected for the set of co-scheduled RBs. For the analysis of this system, we initially need to reformulate the integral in (18). As the MCS selection for the co-scheduled RB set is based on the minimum of SINR realization and the $C()$ function is monotonically increasing, the minimum operator in (II) can be reformulated as: $\min_n\{C(Z_{j,n})|S_{j,n}=1\} = C(\min_n\{Z_{j,n}|S_{j,n}=1\})$. Therefore, the rate expectation of PFS with MCS constraints can be expressed based on the minimum order statistics of the SINR as:

$$\mathcal{R}_j = \sum_{\forall\,\mathcal{A}\in\mathcal{P}(N)} \mathrm{P}\left[S_{j,\mathcal{A}}=1\right] \frac{|\mathcal{A}|}{T_{\mathrm{TTI}}} \int_0^\infty C(z) \cdot f_{Z^{\min}_{j,\mathcal{A}}|S_{j,\mathcal{A}}=1}(z)\ dz \tag{25}$$

where $\mathcal{P}(N)$ is the power set of N (i.e., the set of all subsets $\mathcal{A} = \{n|S_{j,n}(t)=1\}$ of possibly co-scheduled resources) while $|\mathcal{A}|$ is the size of the subset $\mathcal{A}$. The computation of the probability $\mathrm{P}\left[S_{j,\mathcal{A}}=1\right]$ is challenging as the resource blocks in $\mathcal{A}$ are usually subject to correlation due to the delay spread of the propagation environment. We continue our analysis under the assumption that the resource blocks of $\mathcal{A}$ are statistically independent. Furthermore, we assume that the scheduling probability equals that of the channel-state based proportional fair scheduler. Both these simplifications introduce an error that we quantify in the validation part (generally finding that it is negligible). Based on these assumptions, we obtain the scheduling probability of subset $\mathcal{A}$ as:

$$\mathrm{P}\left[S_{j,\mathcal{A}}=1\right] \approx \prod_{n\in\mathcal{A}} \mathrm{P}[S_{j,n}=1] \prod_{m\notin\mathcal{A}} \left(1-\mathrm{P}[S_{j,m}=1]\right). \tag{26}$$

The scheduling probability, i.e., the probability that random variable $\hat{Z}_{j,n}=\frac{Z_{j,n}}{\mathrm{E}[Z_{j,n}]}$ is greater than all other random variables $\hat{Z}$ of the other terminals is given by:

$$\mathrm{P}[S_{j,n}=1] = \mathrm{P}\left[\hat{Z}_{j,n} \geq \max_{\forall g\neq j\in\mathcal{J}} (\hat{Z}_{g,n})\right] \tag{27}$$

$$= \int_0^\infty f_{\hat{Z}_{j,n}}(z) \cdot \prod_{\forall g\neq j\in\mathcal{J}} F_{\hat{Z}_{g,n}}(z)\ \mathrm{d}z \tag{28}$$

$$= \mathrm{E}[Z_{j,n}] \cdot \int_0^\infty \prod_{\forall g\neq j\in\mathcal{J}} F_{Z_{g,n}}(\mathrm{E}[Z_{g,n}]\cdot z) \cdot f_{Z_{j,n}}(\mathrm{E}[Z_{j,n}]\cdot z)\ \mathrm{d}z. \tag{29}$$

The CDF of minimum order statistic $Z_{j,\mathcal{A}}^{\min}$ conditioned to $S_{j,\mathcal{A}}=1$ computes as

$$F_{Z_{j,\mathcal{A}}^{\min}|S_{j,\mathcal{A}}=1}(z) = \mathrm{P}\left[\min_{n\in\mathcal{A}} Z_{j,n} \leq z \,|\, S_{j,\mathcal{A}}=1\right] \tag{30}$$

$$= 1-\mathrm{P}\left[Z_{j,n} > z, \forall\, n\in\mathcal{A} \,|\, S_{j,\mathcal{A}}=1\right] \tag{31}$$

$$= 1-\prod_{n\in\mathcal{A}} \bar{F}_{Z_{j,n}|S_{j,n}=1}(z), \tag{32}$$

in which $\bar{F}_{Z_{j,n}|S_{j,n}=1}(z) = 1-F_{Z_{j,n}|S_{j,n}=1}(z)$. Differentiating yields the PDF

$$f_{Z_{j,\mathcal{A}}^{\min}|S_{j,\mathcal{A}}=1}(z) = \sum_{n\in\mathcal{A}} \Big(f_{Z_{j,n}|S_{j,n}=1}(z) \prod_{\forall\, m\neq n\in\mathcal{A}} \bar{F}_{Z_{j,m}|S_{j,m}=1}(z) \Big). \tag{33}$$

Recalling (8), the probability density function of the scheduled SINR can be expressed as:

$$f_{Z_{j,n}|S_{j,n}=1}(z) = \frac{\prod_{\forall g\neq j\in\mathcal{J}} F_{Z_{g,n}}\left(\frac{\mathrm{E}[Z_{g,n}]}{\mathrm{E}[Z_{j,n}]}\cdot z\right)\cdot f_{Z_{j,n}}(z)}{\mathrm{E}[Z_{j,n}]\cdot \int_0^\infty \prod_{\forall g\neq j\in\mathcal{J}} F_{Z_{g,n}}(\mathrm{E}[Z_{g,n}]\cdot u)\cdot f_{Z_{j,n}}(\mathrm{E}[Z_{j,n}]\cdot u)\ \mathrm{d}u} \tag{34}$$

and the corresponding distribution function is given by:

$$F_{Z_{j,n}|S_{j,n}=1}(z) = \frac{\int_0^z \prod_{\forall g\neq j\in\mathcal{J}} F_{Z_{g,n}}\left(\frac{\mathrm{E}[Z_{g,n}]}{\mathrm{E}[Z_{j,n}]}\cdot y\right)\cdot f_{Z_{j,n}}(y)\ \mathrm{d}y}{\mathrm{E}[Z_{j,n}]\cdot \int_0^\infty \prod_{\forall g\neq j\in\mathcal{J}} F_{Z_{g,n}}(\mathrm{E}[Z_{g,n}]\cdot u)\cdot f_{Z_{j,n}}(\mathrm{E}[Z_{j,n}]\cdot u)\ \mathrm{d}u}. \tag{35}$$

Assuming that each base station applies a homogeneous transmit power to each sub-carrier, the rate expectation for the unique MCS constraint simplifies to:

$$\begin{aligned}\mathcal{R}_j =&\frac{N_{\text{S}}N_{\text{C}}}{T_{\text{TTI}}}\sum_{n=1}^{N}\sum_{m=2}^{M}\binom{n}{N} n\text{P}[S_{j,n}=1]^n\left(1-\text{P}[S_{j,n}=1]\right)^{N-n}c_m \\ &\cdot\int_{z_{m-1}}^{z_m} f_{Z_{j,n}|S_{j,n}=1}(z)\left(1-F_{Z_{j,n}|S_{j,n}=1}\right)^{n-1}(z)\,\text{d}z \qquad (36)\\ =&\frac{N_{\text{S}}N_{\text{C}}}{T_{\text{TTI}}}\sum_{n=1}^{N}\sum_{k=0}^{n-1}\sum_{m=2}^{M}\binom{n-1}{k}\binom{n}{N}(-1)^k n\text{P}[S_{j,n}=1]^n\left(1-\text{P}[S_{j,n}=1]\right)^{N-n}c_m \\ &\cdot\int_{z_{m-1}}^{z_m}\left(F_{Z_{j,n}|S_{j,n}=1}(z)\right)^k\,\text{d}F_{Z_{j,n}|S_{j,n}=1}(z) \qquad (37)\\ =&\frac{N_{\text{S}}N_{\text{C}}}{T_{\text{TTI}}}\sum_{n=1}^{N}\sum_{k=0}^{n-1}\sum_{r=0}^{N-n}\sum_{m=2}^{M}\binom{n-1}{k}\binom{N-n}{r}\binom{n}{N}\frac{(-1)^{k+r}}{k}n\text{P}[S_{j,n}=1]^{n+r-k-1}c_m \\ &\cdot\left(\int_{z_{m-1}}^{z_m}\prod_{\forall g\neq j}F_{Z_{g,n}}\left(\frac{\text{E}[Z_{g,n}]}{\text{E}[Z_{j,n}]}\cdot z\right)f_{Z_{j,n}}(z)\,\text{d}z\right)^{k+1} \qquad (38)\end{aligned}$$

By replacing the solution of the indefinite integral (24) in the last line of (38) and sequentially putting in the limits for the different MCS combinations, we finally obtain the solution of the rate expectation (25) for the practical system model.

C. Asymptotic Analysis Results of Proportional Fair Scheduling in Ultra-Dense Deployments

Finally, we are also interested in asymptotic results especially with respect to the number of interferers. For instance, in ultra-dense cellular network deployments, typically the interference stems from a large number of neighboring base stations as they are deployed with a much higher density than for current cellular networks. As the number of interferers goes to infinity, we are interested in the consequences for the computation of the rate expectation of the PFS algorithm presented in the system model.

Asymptotic analysis: Let the number of interfering base stations I go to infinity and assume that the average received interference power at terminal j from each interfering base station is equally strong [1], i.e., $P_{j,n}/I$. Then, the CDF for an unlimited large number of interfering stations

[1]This is plausible for mainly two scenarios: a) the interfering base stations are relatively far away so that the average interference power can be approximated as equally strong; b) base stations have a clustered spatial positioning. I.e., each BS has a similar distance to the mobile stations.

but with total interference power converging to $P_{j,n}$ is:

$$F_{Z_{j,n}}(z) = \lim_{I\to\infty}\left(1-\prod_{i=1}^{I}\frac{1}{1+\frac{P_{j,n}}{Ip_{0,j,n}}z}\mathrm{e}^{-z\frac{N_0}{p_{0,j,n}}}\right) \tag{39}$$

$$= 1-\frac{1}{\lim_{I\to\infty}\left(1+\frac{1}{I}\frac{P_{j,n}}{p_{0,j,n}}z\right)^{I}}\mathrm{e}^{-z\frac{N_0}{p_{0,j,n}}} \tag{40}$$

$$= 1-\frac{1}{\mathrm{e}^{\frac{P_{j,n}}{p_{0,j,n}}z}}\mathrm{e}^{-z\frac{N_0}{p_{0,j,n}}} \tag{41}$$

$$= 1-\mathrm{e}^{-z\frac{N_0+P_{j,n}}{p_{0,j,n}}}. \tag{42}$$

Note that the SINR distribution converges to an exponential distribution with expectation given by:

$$\mathrm{E}[Z_{j,n}] = \frac{p_{0,j,n}}{N_0+P_{j,n}}. \tag{43}$$

Hence, we obtain:

$$F_{Z_{g,n}}\left(\frac{\mathrm{E}[Z_{g,n}]}{\mathrm{E}[Z_{j,n}]}z\right) = 1-\mathrm{e}^{-z\frac{N_0+P_{g,n}}{p_{0,g,n}}\frac{p_{0,g,n}}{N_0+P_{g,n}}\frac{N_0+P_{j,n}}{p_{0,j,n}}} \tag{44}$$

$$= 1-\mathrm{e}^{-z\frac{N_0+P_{j,n}}{p_{0,j,n}}} \tag{45}$$

$$= F_{Z_{j,n}}(z). \tag{46}$$

Based on these results, the rate integral (10) can be simplified to:

$$\mathcal{R}_j = \frac{N_\mathrm{S}N_\mathrm{C}}{T_\mathrm{TTI}}\sum_{n=1}^{N}\sum_{m=2}^{M}c_m\int_{z_{m-1}}^{z_m}F_{Z_{j,n}}(z)^{|\mathcal{J}-1|}f_{Z_{j,n}}(z)\,\mathrm{d}z \tag{47}$$

$$= \frac{N_\mathrm{S}N_\mathrm{C}}{|\mathcal{J}|T_\mathrm{TTI}}\sum_{n=1}^{N}\sum_{m=2}^{M}c_m\left(F_{Z_{j,n}}^{|\mathcal{J}|}(z_m)-F_{Z_{j,n}}^{|\mathcal{J}|}(z_{m-1})\right). \tag{48}$$

As a consequence, the expected throughput of PFS does not depend on the distribution of the other mobile terminals in the cell any longer. It is based solely on the the average SINR of the mobile terminal under consideration and the total number of terminals in the cell. The same results also apply to noise-limited scenarios with Rayleigh fading of the signal of interest.

Asymptotic analysis for the MCS constraint: The scheduling probability (34) for an unlimited number of interfering base stations and unique MCS can be computed as:

$$F_{Z_{g,n}}(\mathrm{E}[Z_{g,n}]z) = 1-\mathrm{e}^{-z\frac{N_0+P_{g,n}}{p_{0,g,n}}\frac{p_{0,g,n}}{N_0+P_{g,n}}} \tag{49}$$

$$= 1-\mathrm{e}^{-z} \tag{50}$$

and

$$\mathrm{E}[Z_{j,n}] \cdot f_{Z_{j,n}}(\mathrm{E}[Z_{j,n}]z) = \frac{p_{0,j,n}}{N_0 + P_{j,n}} \frac{N_0 + P_{j,n}}{p_{0,j,n}} \mathrm{e}^{-z \frac{N_0+P_{j,n}}{p_{0,j,n}} \frac{p_{0,j,n}}{N_0+P_{j,n}}} \tag{51}$$

$$= \mathrm{e}^{-z}. \tag{52}$$

Replacing (50) and (52) in (29) the scheduling probability is simplified to:

$$\mathrm{P}[S_{j,n} = 1] = \int_0^{\infty} (1 - \mathrm{e}^{-z})^{|\mathcal{J}-1|} \mathrm{e}^{-z} \,\mathrm{d}z \tag{53}$$

$$= \frac{1}{|\mathcal{J}|}. \tag{54}$$

The insight from this expression is that for ultra-dense or noise-limited scenarios, the scheduling probability depends only on the number of terminals in the cell. Therefore, each mobile terminal has the same scheduling probability.

The scheduled SINR PDF in (34) is simplified to:

$$f_{Z_{j,n}|S_{j,n}=1}(z) = |\mathcal{J}| F_{Z_{j,n}}^{|\mathcal{J}|-1}(z) f_{Z_{j,n}}(z) \tag{55}$$

and the corresponding CDF can be rewritten as:

$$F_{Z_{j,n}|S_{j,n}=1}(z) = |\mathcal{J}| \int_0^{z} F_{Z_{j,n}}^{|\mathcal{J}|-1}(y) f_{Z_{j,n}}(y) \,\mathrm{d}y \tag{56}$$

$$= |\mathcal{J}| F_{Z_{j,n}}^{|\mathcal{J}|}(z). \tag{57}$$

The expectation of the scheduled SINR can be computed as:

$$\begin{aligned}
\mathrm{E}[Z_{j,n}|S_{j,n} = 1] &= |\mathcal{J}| \frac{N_0 + \bar{\eta}_{j,n}}{\bar{p}_{0,j,n}} \int_0^{\infty} z \left(1 - \mathrm{e}^{-z \frac{N_0+\bar{\eta}_{j,n}}{\bar{p}_{0,j,n}}}\right)^{|\mathcal{J}|-1} \mathrm{e}^{-z \frac{N_0+\bar{\eta}_{j,n}}{\bar{p}_{0,j,n}}} \,\mathrm{d}z \\
&= |\mathcal{J}| \frac{N_0 + \bar{\eta}_{j,n}}{\bar{p}_{0,j,n}} \sum_{k=0}^{\mathcal{J}-1} \binom{|\mathcal{J}|-1}{k} \int_0^{\infty} z \mathrm{e}^{-z \frac{(N_0+\bar{\eta}_{j,n})(k+1)}{\bar{p}_{0,j,n}}} \,\mathrm{d}z \\
&= \sum_{k=0}^{\mathcal{J}-1} \binom{|\mathcal{J}|-1}{k} \frac{\bar{p}_{0,j,n}}{N_0 + \bar{\eta}_{j,n}} \frac{(-1)^k |\mathcal{J}|}{(k+1)^2}.
\end{aligned} \tag{58}$$

We can also determine the gain of the scheduled SINR compared to a channel-agnostic scheduler like round-robin which is given by:

$$G = \frac{\mathrm{E}[Z_{j,n}|S_{j,n} = 1]}{\mathrm{E}[Z_{j,n}]} \tag{59}$$

$$= \sum_{k=0}^{\mathcal{J}-1} \binom{|\mathcal{J}|-1}{k} \frac{(-1)^k |\mathcal{J}|}{(k+1)^2}. \tag{60}$$

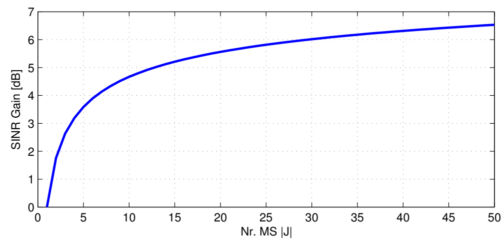


Fig. 1: SINR gain of PFS scheduling with respect to the number of mobile stations per cell.

Here we notice, that the SINR gain due to PFS for the asymptotic and noise limited scenarios is independent of the location of terminals and depends only on the number of terminals present in the cell. A plot of the SINR gain according to Equation (60) with respect to the number of mobile stations per cell is given in Figure 1. As it can be noticed, the highest gain gradient is obtained for relatively small numbers (up to 10) of mobile stations. Whereas for a higher concentration of MS per cell the gain gradient decreases and larger concentrations of MSs per cell (more than 20 MS) cause a marginal SINR gain. Replacing in the rate expression (38) the identities from (54), (55) and (57) we finally obtain:

$$\mathcal{R}_j = \frac{N_\text{S} N_\text{C}}{T_\text{TTI}} \sum_{n=1}^{N} \sum_{m=2}^{M} n \binom{n}{N} |\mathcal{J}|^{-n} \left(1 - \frac{1}{|\mathcal{J}|}\right)^{N-n} |\mathcal{J}| c_m \cdot \int_{z_{m-1}}^{z_m} F_{Z_{j,n}}^{|\mathcal{J}|-1}(z) f_{Z_{j,n}}(z) \left(1 - |\mathcal{J}| F_{Z_j}^{|\mathcal{J}|}(z)\right)^{n-1} \mathrm{d}z \tag{61}$$

$$= \frac{N_\text{S} N_\text{C}}{T_\text{TTI}} \sum_{n=1}^{N} \sum_{m=2}^{M} n \binom{n}{N} |\mathcal{J}|^{-n} \left(1 - \frac{1}{|\mathcal{J}|}\right)^{N-n} |\mathcal{J}| c_m \cdot \int_{z_{m-1}}^{z_m} F_{Z_{j,n}}^{|\mathcal{J}|-1}(z) \left(1 - |\mathcal{J}| F_{Z_{j,n}}^{|\mathcal{J}|}(z)\right)^{n-1} \mathrm{d}F_{Z_{j,n}}(z) \tag{62}$$

$$= \frac{N_\text{S} N_\text{C}}{T_\text{TTI}} \sum_{n=1}^{N} \sum_{m=2}^{M} \sum_{k=0}^{n-1} n \binom{n-1}{k} \binom{n}{N} (-1)^k |\mathcal{J}|^k |\mathcal{J}|^{-n} \left(1 - \frac{1}{|\mathcal{J}|}\right)^{N-n} |\mathcal{J}| c_m \cdot \int_{z_{m-1}}^{z_m} F_{Z_{j,n}}^{|\mathcal{J}|+k-1}(z)\, \mathrm{d}F_{Z_{j,n}}(z). \tag{63}$$

$$= \frac{N_\text{S} N_\text{C}}{T_\text{TTI}} \sum_{m=2}^{M} \sum_{n=1}^{N} \sum_{k=0}^{n-1} n \binom{n-1}{k} \binom{n}{N} (-1)^k |\mathcal{J}|^{k+1-N} \left(1 - |\mathcal{J}|\right)^{N-n} \frac{1}{|\mathcal{J}|+k} c_m \cdot \left(F_{Z_{j,n}}^{|\mathcal{J}|+k}(z_m) - F_{Z_{j,n}}^{|\mathcal{J}|+k}(z_{m-1})\right). \tag{64}$$

Also for a system model with unique MCS constraints, we hence showed that in ultra-dense deployments the throughput expectation depends only on the average SINR of the corresponding terminal and the total number of terminals in the cell. It does not depend on the location of the other terminals.

V. Numerical Evaluation

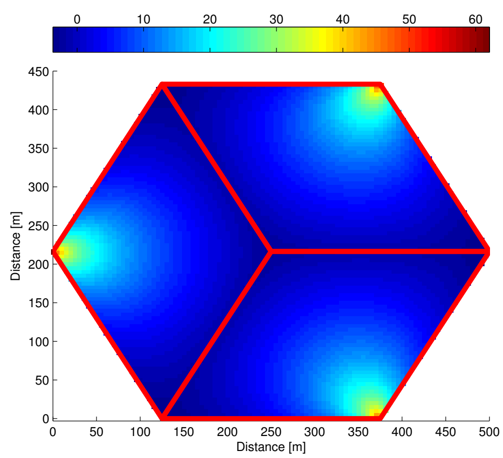
In this section we validate the mathematical derivations presented so far by means of simulations. First, we present the simulation parameters and discuss our evaluation methodology. Then, we proceed with the validation of the scheduled SINR and throughput expectation models. Finally, we benchmark our model in comparison to state-of-the-art models and also quantify the (still remaining) error of our models in comparison to simulation values.

A. Simulation Parameters

We consider in our evaluation the downlink communication of an urban deployment presented in Figure 2. Each cell, whose borders are marked by red lines, hosts a set of $|\mathcal{J}| = 10$ mobile stations uniformly distributed in the cell area. The communication is limited by co-channel interference generated from two sets of interfering base stations. The (respectively) closer two base stations use 120^o sectorized antennas with 15 dBi maximal gain meanwhile the ones deployed in the outer ring use omnidirectional antennas. The centre frequency and bandwidth are set respectively to $f_{\text{C}} = 1.9$ GHz and $B = 5$ MHz, which results for LTE in $N = 25$ resource blocks. Each resource block transmits $N_{\text{S}} = 14$ symbols in time while it contains $N_C = 12$ subsequent subcarriers in frequency. The transmission power per base station equals 20 W while we assume a frequency reuse of one (i.e., all subcarriers are interfered equally). Transmission slot durations are set to $T_{\text{TTI}} = 1$ms and the window size W of PFS is set to 1000 TTIs. Fast fading realizations are generated according to Jake's model generating time correlated channel gains.

We use the COST-231 Hata and Walfish Ikegami models to model the pathloss in a cellular communication system. The former is used the simulation of suburban (SM) and urban macro (UMa) cell scenarios, whereas the latter is used for the simulation of urban microcell (UMi) environments. The corresponding cell-radii of the UMa, UMi and SM scenarios are 100, 250 and 650 m. The shadowing parameter σ_{SH} is scenario specific and takes values 4, 6 and 8 dB respectively for urban micro, urban macro and suburban scenarios [27]. Note that our models can be applied as long as shadowing is considered to be a constant, i.e. if the shadowing coefficient changes significantly, the rate expectations need to be recomputed. For each scenario in total 30 different, randomly drops of all terminals are simulated to assure statistical confidence.

We work here with non-line-of sight communications which is commonly used for urban scenarios. We assume isotropic scattering with no dominant path present at the receiver. The model used to simulate the fast-fading gain is based on Jake's model [28] and is implemented in [29]. It takes into account the Doppler spread for time selectivity as well as the delay spread for frequency selectivity. Correlation in time is simulated by summing up a set of sinusoids. For the three considered scenarios in our work, the mean delay spread for the scenarios UMi, UMa and SM takes respectively values 43 ns, 357 ns and 991 ns as recommended in [30]. Whereas, the mobility of the background scatterers for each of the three scenarios UMi, UMa and SM takes the corresponding values of 3 Kmh, 30 Kmh and 60 Kmh.



(a) SINR distribution in the UMa scenario.

Fig. 2: SINR distribution for the UMa scenario.

B. Evaluation Methodology

The validation is performed for the set of base stations located in the corners of the central hexagonal cluster in Figure 2. Note that in this configuration, the corresponding two other base stations serve as interferers. Firstly, we validate the SINR CDFs ((71) and (35)), which are important expressions of the throughput expectation models. The evaluation is performed for two extreme locations of terminals that might arise in a cell: cell core (MS A) and cell edge (MS B). Then, we evaluate the throughput expectation models for both the ideal and practical system models. For this, the relative error ϵ_j between the throughput expectation models and simulations is used. It is defined as:

$$\epsilon_j = \frac{|\mathcal{R}_j - \mathcal{R'}_j|}{\mathcal{R'}_j} \cdot 100, \tag{65}$$

where $\mathcal{R'}_j$ and $\mathcal{R}_j$ are respectively the rates obtained from system level simulations and from the theoretical models developed in this paper as well as from related work discussed in Section III. The value of ϵ_j is colour-coded and represented spatially by displaying it at the respective position of the terminal j in Figure 3. This performance metric is used to quantify the accuracy of the proposed model as well as the models from the literature: a) interference as noise approximation using the PDF (11) for the rate expectation model in (10), b) Gaussian approximation (15), c) i.i.d. priority approximation (16); d) simplistic model (17); as well as e) unique MCS and interference as noise model obtained by replacing the exact SINR distribution functions in (25) with the interference as noise distribution (11). Simulations are conducted according to an Omnet++ based LTE model, which performs proportional fair scheduling based on the algorithm introduced in the system model section.

In LTE systems the averaging is performed over the rates and not over the SINR. However, the analysis of PF scheduling based directly on the probability density function (PDF) of the rate results challenging. In fact, in literature two main approaches are used to describe the long-term performance of PF scheduling. The first one approximates the instantaneous rate distribution to a Gaussian distribution function [20], [21], [5], [22]. Whereas, the second approach analyses the performance of a channel-state-based (i.e. SNR) PF scheduling (performing the averaging over the SINR) [10], [11], [7], [6], [14], [8]. In our paper we use the latter approach by including in the analysis also the impact of the interference. We use then the obtained model to describe the performance of a rate-based PF scheduler. This approximation has also an implication on how we validate the models. Firstly, we validate the model for the assumptions taken in this paper, SINR-based PFS with exponentially distributed channel gains. Then, we extend the analysis for practical deployments operating on rate-based PFS in time and frequency correlated environments.

C. Results

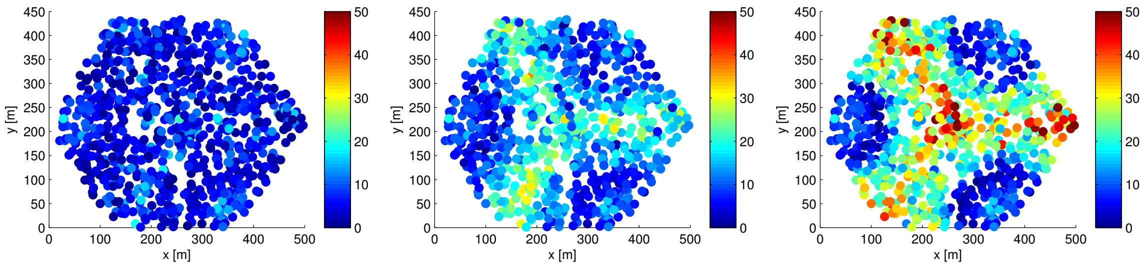
The ECDF of the relative error distribution for all the considered scenarios is presented in Figure 4. In order to observe the error distribution among a cell we also plot in Figure 3 the relative error of the mobile stations with respect to their position. Although the plots are only for the UMa scenario we notice similar patterns also for the other scenarios as well which we skip due to space limitations.

In general, for the SINR-based PFS we notice a good match between simulations and our derived throughput expectation model. The remaining relative error is due to the approximation of the average SINR over e limited window size W by the expectation of the SINR. The accuracy of the other state-of-the art models is significantly lower, and worst for the simplistic model. Regarding the terminal positions, the lowest accuracy occurs for the cell-edge located terminals. For the other models this inaccuracy stems from the assumptions on the basic distribution of the SINR, rate and the expectations ($\mathrm{E}[Z_{j,n}]$ and $\mu_{R_{j,n}}$). The inaccuracy of the simplistic model on the other side stems from neglecting the maximum order statistics effect due to the scheduling gain of PFS.

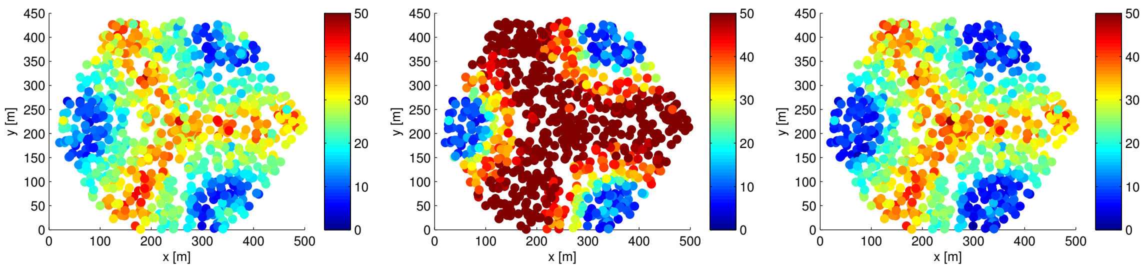
Although, the *unique MCS and interference as noise* model captures most of the system aspects it still provides a relatively low accuracy in comparison to the model that we propose in this paper. By comparing these two models it is obvious that ignoring the interference variation in the analysis leads to considerable accuracy degradation. Especially, inaccuracies stemming from the SINR distribution approximation are emphasized when considering the joint SINR distribution due to the MCS constraint (e.g. effective SINR or minimum order statistics) is considered.

We extend the analysis of our model for rate-based PFS. We compare the approximation accuracy of the SINR based PFS introduced here with that of a rate-based PFS in time and correlated environments. The results are shown in Figure 5. We notice that even for rate-based PF scheduling the channel-state based model is still a better approximation that the state of-the-art. Especially for interference limited scenarios the accuracy of our model is considerably better than approximations used so far.

Finally, in Figure 6 we also present the SINR distributions for cell edge and cell center terminals in an SINR-based PFS operating in time correlated channel environments. In our analysis there are two factors which can lead to a mismatch between simulations and the analysis,

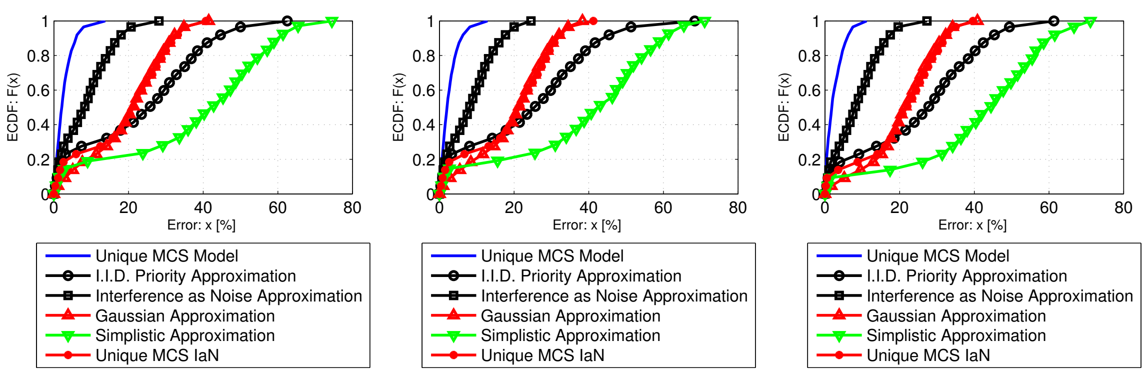


(a) Relative error for the proposed throughput expectation model. (b) Relative error for the interference as noise model. (c) Relative error for the i.i.d. priority approximation model.



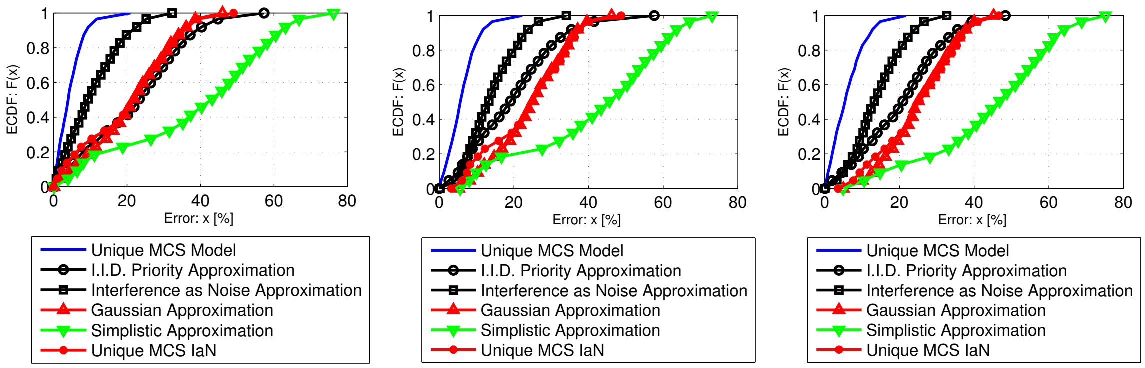
(d) Relative error for the Gaussian approximation model. (e) Relative error for the simplistic approximation model. (f) Relative error for unique MCS and interference as noise model model.

Fig. 3: Error distribution for the UMa scenario. Similar results were observed also for the other scenarios. The color codes represent the relative error between the simulations and the corresponding expectation model.



(a) Error ECDF for the UMi scenario. (b) Error ECDF for the UMa scenario. (c) Error ECDF for the SM scenario.

Fig. 4: Error ECDF in the simulation area for the SINR-based PFS.



(a) Error ECDF for the UMi scenario. (b) Error ECDF for the UMa scenario. (c) Error ECDF for the SM scenario.

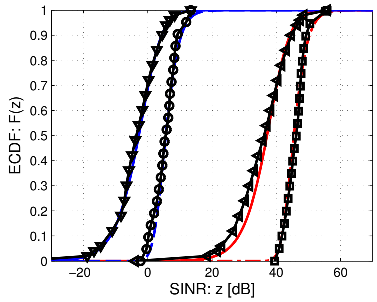
Fig. 5: Error ECDF in the simulation area for the rate-based PFS.

namely the correlation in time and the limited window size W. However, as we can notice from Figure 6, these factors combined, do not have a noticeable impact on the distributions. Additionally, we notice a similar SINR gain irrespective of the MS position. This observation matches with the analytical results for an unlimited amount of interfering base stations as well as for noise-limited systems. We showed by (60) that the gain is independent of the location of terminals.

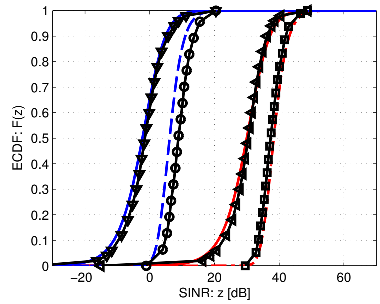
VI. Conclusions

In this paper we introduced closed-form expressions for the expected throughput of proportional fair scheduling in interference limited scenarios. Accurate distribution of the SINR was considered leading to challenging integral expressions to be solved. Nevertheless, the obtained closed-form solutions lead to a higher accuracy than state-of-the-art throughput expectation models even for time and frequency correlated channel gains. We point out that accurate SINR distributions are required for the analysis of OFDMA systems operating on common MCS.

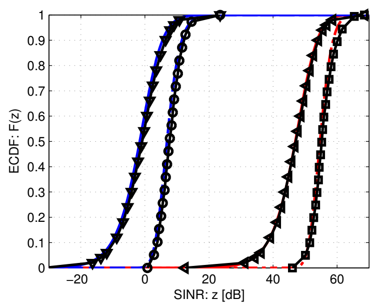
The analytical models are further extended also for ultra-dense deployments leading to several insights. Analytically we showed that for such scenarios the SINR distribution converges to an exponential one. Additionally, the SINR gain due to PFS and scheduling probability depends only on the total number of mobile stations per cell. Finally, the throughput expectations for such scenarios is independent of the location of the other terminals in the cell.



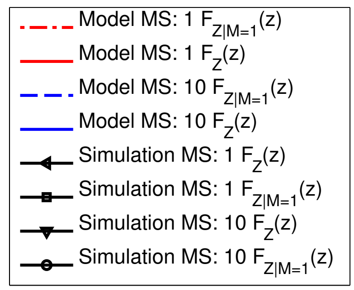
(a) SINR distribution for the UMi scenario.



(b) SINR distribution for the UMa scenario.



(c) SINR distribution for the SM scenario.



(d) Legend

Fig. 6: SINR transformation due to proportional fair scheduling for terminals located in cell center and cell edge.

Appendix A
Computation of the SINR distribution

$$\mathrm{P}\left(\frac{X_{0,j,n}\bar{p}_{0,j,n}}{\sum_{i=1}^{I} X_{i,j,n}\bar{p}_{i,j,n} + N_\mathrm{o}} > z\right) = \mathrm{P}\left(X_0 > z \cdot \left(\sum_{i=1}^{I} \frac{\bar{p}_{i,j,n}}{\bar{p}_{0,j,n}} X_i + \frac{N_\mathrm{o}}{\bar{p}_{0,j,n}}\right)\right) \tag{66}$$

$$= \mathrm{P}\left(X_{0,j,n} - \sum_{i=1}^{I} \frac{\bar{p}_{i,j,n}}{\bar{p}_{0,j,n}} X_{i,j,n} z - \frac{N_\mathrm{o}}{\bar{p}_{0,j,n}} z > 0\right) \tag{67}$$

$$= \int_{t_1=0}^{\infty} \cdots \int_{t_I=0}^{\infty} \mathrm{e}^{-(t_1+\ldots+t_I)} \prod_{i=1}^{I} \frac{\bar{p}_{0,j,n}}{\bar{p}_{i,j,n} z} \mathrm{e}^{-\frac{\bar{p}_{0,j,n}}{z\bar{p}_{i,j,n}} t_i} \mathrm{e}^{-z\frac{N_\mathrm{o}}{\bar{p}_{0,j,n}}}\, \mathrm{d}t_1 \cdots \mathrm{d}t_I \tag{68}$$

$$= \prod_{i=1}^{I} \int_{t_i=0}^{\infty} \frac{\bar{p}_{0,j,n}}{\bar{p}_{i,j,n} z} \mathrm{e}^{-\left(1+\frac{\bar{p}_{0,j,n}}{\bar{p}_{i,j,n} z}\right) t_i} \mathrm{e}^{-z\frac{N_\mathrm{o}}{\bar{p}_{0,j,n}}}\, \mathrm{d}t_i \tag{69}$$

$$= \prod_{i=1}^{I} \frac{1}{1 + \frac{\bar{p}_{i,j,n}}{\bar{p}_{0,j,n}} z} \mathrm{e}^{-z\frac{N_\mathrm{o}}{\bar{p}_{0,j,n}}} \tag{70}$$

The PDF transformation of the sum of random variables in (67) is computed from their convolution (see [26]), which we apply in (68). As the interfering sources are independent of each other, we can transform Expression (68) (with the multiple integrations) to the product of independent integrals (69). Finally, subtracting (70) by one yields the SINR CDF:

$$F_{Z_{j,n}}(z) = 1 - \prod_{i=1}^{I} \frac{1}{1 + \frac{p_{i,j,n}}{p_{0,j,n}} z} \mathrm{e}^{-z \frac{N_0}{p_{0,j,n}}}. \tag{71}$$

We are interested in bringing the CDF function to a sum of elementary functions. Therefore, we decompose the CDF (71) by means of partial fraction decomposition using the following corollary:

Corollary 1 [31, pg. 278]: *Let $p(x)$ be a polynomial of degree p and $q(x)$ be a polynomial with distinct roots α_i and degree n, where $p < n$. Then, the ratio of these polynomials can be decomposed into partial fractions as:*

$$\frac{p(x)}{q(x)} = \sum_{i=1}^{n} \frac{p(\alpha_i)}{q'(\alpha_i)} \cdot \frac{1}{x - \alpha_i} \tag{72}$$

where $q'(x)$ is the first derivative of the polynomial $q(x)$.

Reorganizing (71) and applying Corollary 1, the decomposed form of CDF is given in (19).

Appendix B
Solution of the Indefinite Integral

The CDF product decomposition is based on the following corollary:

Corollary 2 [32, pg. 1]: *Let $r_1, \ldots, r_n$ be roots of the polynomial $\prod_{g=1}^{n}(x - r_g)$. Then the monic polynomial expands as: $\prod_{g=1}^{n}(x - r_g) = \sum_{k=0}^{n}(-1)^k \sigma_k x^{n-k}$ where the coefficients σ_k are the elementary symmetric functions of $r_1, \ldots, r_n$,*

$$\sigma_k = \sum_{g_1 < g_2 < \cdots < g_k} \prod_{l=1}^{k} r_{g_l}. \tag{73}$$

Note the special cases $\sigma_0 = 1$ and $\sigma_k = 0$ for $j > n$. For example, if $n = 4$ then the nonzero elementary functions σ_k are given by the permutations:

$$\sigma_0 = 1,$$

$$\sigma_1 = r_1 + r_2 + r_3 + r_4,$$

$$\sigma_2 = r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4,$$

$$\sigma_3 = r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4,$$

$$\sigma_4 = r_1 r_2 r_3 r_4$$

We now apply fraction decomposition and polynomial expansion to the product of the CDFs within the integral of (18). For readability, we concentrate on an arbitrary resource block and omit in the following the index n. Applying Corollary 2 and repetitively using partial fraction decomposition according to Corollary 1, we expand the product of CDFs as:

$$\prod_{\forall g \neq j} F_{Z_g}\left(\frac{\mathrm{E}[Z_g]}{\mathrm{E}[Z_j]} \cdot z\right) = \prod_{\forall g \neq j}\left(1 - \sum_{i=1}^{I} U(\hat{c}_{i,g}) \frac{1}{\hat{c}_{i,g} + z} \mathrm{e}^{-z\hat{c}_{0,g}}\right) \tag{74}$$

$$= 1 + \sum_{k=1}^{|\mathcal{J}|-1} (-1)^k \sum_{g_1 < g_2 < \cdots < g_k} \prod_{l=1}^{k} \sum_{i=1}^{I} U(\hat{c}_{i,g_l}) \frac{1}{\hat{c}_{i,g_l} + z} \mathrm{e}^{-z\sum_{l=1}^{k} \hat{c}_{0,g_l}} \tag{75}$$

$$= 1 + \sum_{k=1}^{|\mathcal{J}|-1} (-1)^k \sum_{g_1 < g_2 < \cdots < g_k} \sum_{r=1}^{k} \sum_{i_r=1}^{I} \prod_{l=1}^{k} U(\hat{c}_{i_r,g_l}) \frac{1}{\hat{c}_{i_r,g_l} + z} \mathrm{e}^{-z\sum_{l=1}^{k} \hat{c}_{0,g_l}} \tag{76}$$

$$= 1 + \sum_{k=1}^{|\mathcal{J}|-1} (-1)^k \sum_{g_1 < g_2 < \cdots < g_k} \sum_{r=1}^{k} \sum_{i_r=1}^{I} \sum_{l=1}^{k} W(\hat{c}_{i_r,g_l}) \frac{1}{\hat{c}_{i_r,g_l} + z} \mathrm{e}^{-z\sum_{l=1}^{k} \hat{c}_{0,g_l}} \tag{77}$$

where $\hat{c}_{i,g} = \frac{p_{0,g}}{p_{i,g}} \frac{\mathrm{E}[Z_j]}{\mathrm{E}[Z_g]}$, $\hat{c}_{0,g} = \frac{N_{\mathrm{o}}}{p_{0,g}} \frac{\mathrm{E}[Z_g]}{\mathrm{E}[Z_j]}$, and $W(\hat{c}_{i_r,g_l}) = \prod_{m=1}^{k} U(\hat{c}_{i_r,g_m}) \left(\sum_{l=1}^{k} \prod_{q \neq l} (\hat{c}_{i_r,g_q} - \hat{c}_{i_r,g_l})\right)^{-1}$.

By multiplying the expanded CDF product with the corresponding PDF and applying partial fraction decomposition we obtain the following expression :

$$\int \prod_{\forall g \neq j} F_{Z_g}\left(\frac{\mathrm{E}[Z_g]}{\mathrm{E}[Z_j]} \cdot z\right) f_{Z_j}(z)\,\mathrm{d}z = F_{Z_j}(z) + \sum_{k=1}^{|\mathcal{J}|-1} (-1)^k \sum_{g_1<g_2<\cdots<g_k} \sum_{r=1}^{k} \sum_{i_r=1}^{I} \sum_{l=1}^{k} \sum_{t=1}^{I}$$
$$W(\hat{c}_{i_r,g_l}) U(c_{t,j}) \int \left(\frac{1}{\hat{c}_{i_r,g_l}+z}\frac{1}{(c_{t,j}+z)^2} + \frac{1}{\hat{c}_{i_r,g_l}+z}\frac{c_{0,j}}{c_{t,j}+z}\right) \mathrm{e}^{-z\sum_{l=1}^{k}\hat{c}_{0,g_l}+c_{0,j}}\,\mathrm{d}z \tag{78}$$

$$= F_{Z_j}(z) + \sum_{k=1}^{|\mathcal{J}|-1} (-1)^k \sum_{g_1<g_2<\cdots<g_k} \sum_{r=1}^{k} \sum_{i_r=1}^{I} \sum_{l=1}^{k} \sum_{t=1}^{I} W(\hat{c}_{i_r,g_l}) U(c_{t,j})$$
$$\left((A-C)\int \frac{\mathrm{e}^{-zD}}{\hat{c}_{i_r,g_l}+z}\,\mathrm{d}z + (B+C)\int \frac{\mathrm{e}^{-zD}}{c_{t,j}+z}\,\mathrm{d}z - A\int \frac{\mathrm{e}^{-zD}}{(c_{t,j}+z)^2}\,\mathrm{d}z\right) \tag{79}$$

Finally, the solution of the elementary integrals is known and the final solution is given in (18).

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