Statistical Analysis of OFDMA Assignments

Maria Vega Martin [1], James Gross [2]

[1] *Ericsson Local Support*
Ericsson España, S.A
maria.vega.martin@ericsson.com

[2] *Mobile Network Performance Group*
UMIC Research Centre
RWTH Aachen University, Germany
gross@umic.rwth-aachen.de

***Abstract*—In this paper we investigate statistical properties of IP optimal subcarrier assignments in the down-link of an OFDMA cell. Essentially, we show that such subcarrier assignments do have "hidden" statistical properties which can be exploited to design an extremely simple assignment strategy. We investigate such statistical properties and the corresponding performance of the simple assignment rules regarding a specific, well-known, NP-hard assignment problem. Although our approach is "light-weight" in complexity, the strategy provides a significant performance gain compared to static assignments. Finally, we investigate our approach also for other optimization problems in the context of multi-user OFDMA assignments.**

I. Introduction

During the last years the orthogonal frequency division multiplexing (OFDM) transmission scheme has become very popular, especially among standardization groups. While OFDM is applied today in high-rate PHY extensions of IEEE 802.11, it is considered as strong candidate for next generation wireless systems in general (not to mention the application of OFDM in IEEE 802.16 systems). As mentioned above, OFDM primarily features beneficial PHY characteristics – making efficient and cheap communication technology possible in broadband wireless channels – but it also offers potentials regarding higher layers, often considered in the context of cross-layer adaptation [1].

Among these potentials, channel (i.e. subcarrier) dependent multi-user scheduling is one of the most often considered research issues of the last years [2], [3], [4]. The underlying rationale for this approach is simple: Due to the inherent diversity (in time as well as frequency) among different terminals and subcarriers in a multi-user OFDM system, there are always sets of subcarriers with a particularly low attenuation for any observed terminal. Furthermore, those subcarriers which have a rather high attenuation (due to fading) are likely to have a rather low attenuation to some other terminal. Hence, resource allocation strategies in such channel dependent multi-user OFDM systems aim at assigning terminals disjoint sets of subcarriers such that the diversity among the subchannel attenuations is exploited to a maximum (usually mathematically described by an optimization model). Such channel-dependent resource allocation strategies achieve a superior performance compared to any static resource allocation strategy. However, several issues remain to be solved prior to the practical application. One very important of them relates to the complexity (and hence the run-time characteristic) of the assignment algorithm. Potentially, the assignment problem can be NP-hard [5] requiring thus a suboptimal assignment algorithm in practice. Still, run times of the algorithms are crucial as the quality of the channel information (the information at the base station / access point about the channel attenuations) degrades over time and apart from the run-time of the algorithm other factors add to the total time spread between collecting the channel information and adapting data transmissions to the current channel states. Thus, the run time of algorithms is still open today and remains an area of ongoing research.

In this paper we are interested in extremely simple assignment schemes targeting at the low end of future OFDM systems (like future wireless local area networks). We pick one of the hardest optimization objectives and analyze then a large set of optimal assignments in a statistical manner. From the observations we deduce afterwards assignment rules and evaluate their performance. In fact we show that this statistical approach provides a significant performance improvement while having a very low complexity.

II. System Model

We consider the following system model: A cell of a wireless system serves J terminals. Time is slotted into frames indexed by i, each frame is split into a down-link and an up-link phase. We only consider the down-link transmission direction. Data transmission in the cell is organized by the base station which assigns transmission resources to terminals for the down-link direction at the beginning of each frame. The system employs OFDM for data transmission in the down-link. We assume a total bandwidth of B hertz to be split into N subcarriers. Each subcarrier n is subject to random channel variations due to path loss, shadowing and fading. However, we assume the frames to be short enough such that the channel quality of a subcarrier stays constant during each frame. Denote the coefficient of subcarrier n at frame i to terminal j as $h_{j,n}^{(i)}$. Given the transmit power applied to each subcarrier $p_n^{(i)}$ the resulting signal-to-noise ratio (SNR) is obtained by $\gamma_{j,n}^{(i)} = p_n^{(i)} \cdot \left(h_{j,n}^{(i)}\right)^2 / n_0$, where n_0 denotes the total noise power per subcarrier. In the following we assume a static and equal power distribution over all subcarriers (i.e. $p_n^{(i)} = P_{\text{sub}} : \forall n, i$). Depending on the instantaneous SNR adaptive modulation and coding is applied. The corresponding "effective" bit amount that can be transmitted on subcarrier n at frame time i to terminal j is denoted by $b_{j,n}^{(i)}$ and is a non-decreasing function of the SNR $b_{j,n}^{(i)} = F\left(\gamma_{j,n}^{(i)}\right)$. At the beginning of each frame we assume the base station to have the complete knowledge of the channel states h and therefore of the SNRs γ which ultimately is transformed into a so called "bit matrix" $\mathbf{B}^{(i)} = \left(b_{j,n}^{(i)}\right)$. Depending on that bit matrix the base station assigns different subsets of subcarriers to terminals, determined by the binary assignment variable $x_{j,n}^{(i)}$. In this study we assume that the base station constantly tries to maximize the minimum amount of throughput among all scheduled terminals J. That is, the base station tries to solve the following optimization problem (referred to as rate-adaptive optimization problem [3]) for each frame:

$$\begin{aligned} \max \quad & \epsilon \\ \text{s.t.} \quad & \sum_{\forall j} x_{j,n}^{(i)} \leq 1 \quad \forall n \\ & \sum_{\forall n} b_{j,n}^{(i)} \cdot x_{j,n}^{(i)} \geq \epsilon \quad \forall j \end{aligned} \tag{1}$$

This optimization approach applies for example if all terminal queues at the base station are heavily backlogged with bulk data and they should all be served equally fast. Optimization problem 1 has been shown to be NP hard and in fact practical instances can be very hard to solve (to optimality) [5] despite the relatively small size of the IP.

III. Statistical Approach

We are interested in finding simple assignment rules which can still provide a better performance than assigning subcarriers without utilizing the channel knowledge either in an FDM or TDM fashion (i.e. as in the case of IEEE 802.11a/g, where – after the acquisition of the channel – all subcarriers are used by the corresponding station). In order to find such assignment rules, we take a statistical approach. In general we observe a set of optimal solutions and analyze these in order to come up with a heuristic. Let us start by generating a series of matrices $\mathbf{B}$ and solving each offline to optimality by an IP solver (in this case CPLEX [6]). We end up with a series of binary assignment matrices $\mathbf{X}$. As first approach we generate from the series of $\mathbf{X}$ matrices the corresponding histogram of assigning any pair of subcarrier n and terminal j (under the regime given by the optimization problem 1 and the chosen IP solver). Formally, we consider the assignment probability given by $p_{j,n} = \sum_i x_{j,n}^{(i)}/I$, where I is the total number of "input" matrices $\mathbf{X}$ considered. We compute this assignment probability for a scenario corresponding to a wireless local area network: $J = 8$ terminals, $N = 48$ subcarriers, 4 modulation types (BPSK, QPSK, 16QAM and 64QAM), bandwidth of $B = 16.25\text{MHz}$, a target bit error rate of 10^{-2}, path loss exponent of $\alpha = 2.4$, shadowing (log-normal distributed) variance of $\sigma^2 = 5.8$ dB, delay spread of 0.15 μsec., frame length of 2 msec., center frequency of 5.2 GHz, total transmit power of 10 mW, maximum Doppler shift of 18 Hz, fixed equidistant terminal positions along the cells radius, cell radius of 100 m. The resulting assignment probability matrix for $I = 5000$ consecutive frames is shown in Figure 1. In Figure 1 we observe basically that there

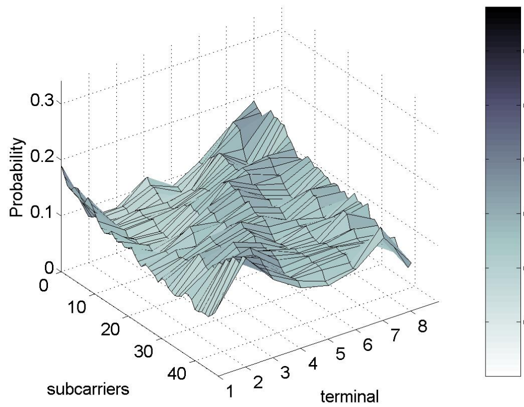

Fig. 1. Assignment probability matrix for cell size of 100 m.

is no significant assignment probability (i.e. above average probability) that could be used. In fact, there is also no reason for any such significant probability. Thus, any statistical assignment based on the histogram will provide the same performance as a static assignment (i.e. without usage of the channel knowledge). Note that this is the case for any observed probability matrix that we have investigated.

Next, we consider the same data (same set of $\mathbf{B}$ and $\mathbf{X}$ matrices), however, we introduce a specific processing. Let us pick a certain $\mathbf{B}^{(i)}$ matrix with its corresponding optimal assignment $\mathbf{X}^{(i)}$. Next, determine the row and column weights of $\mathbf{B}^{(i)}$ (thus, considering the sets of $\mathbf{r^{(i)}} := \{\forall j : \sum_n b_{j,n}^{(i)}\}$ and $\mathbf{c^{(i)}} := \{\forall n : \sum_j b_{j,n}^{(i)}\}$). Then we sort in descending order the two sets $\mathbf{c^{(i)}}$ and $\mathbf{r^{(i)}}$ and modify $\mathbf{B}^{(i)}$ accordingly (i.e. exchanging rows and columns). Let us denote the rearranged matrix of $\mathbf{B}^{(i)}$ by $\mathfrak{B}^{(i)}$. Intuitively, this rearrangement puts the subcarriers with the highest overall weight (which are in a good state with respect to a lot of different terminals) to the top while the terminals with the most subcarriers in a good state are put to the left. Given the optimal assignment $\mathbf{X}^{(i)}$ of $\mathbf{B}^{(i)}$, we obtain the corresponding optimal assignment of $\mathfrak{B}^{(i)}$ by performing the same rearrangement to $\mathbf{X}^{(i)}$, which yields $\mathfrak{X}^{(i)}$. Next, we redo the histogram matrix, i.e. based on the I matrices $\mathfrak{X}^{(i)}$ we recalculate the assignment probabilities given by $\mathfrak{p}_{j,n} = \sum_i \mathfrak{x}_{j,n}^{(i)}/I$. The corresponding result (for the same parameterization as above) is shown in Figure 2. This ordered probability matrix $\mathfrak{P}_I$ owns some interesting properties. First of all, it gives particular importance to the "weakest" terminals as they are assigned the "statistically"

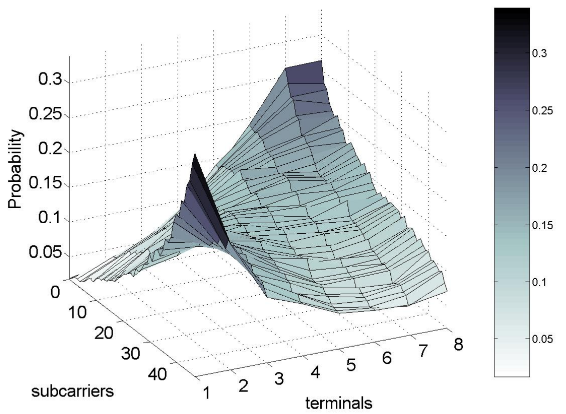

Fig. 2. Ordered assignment probability matrix for cell size of 100 m.

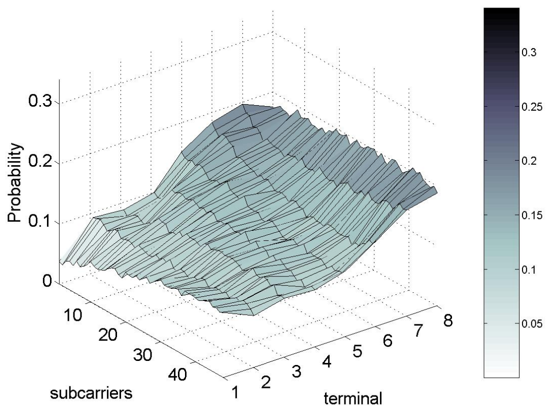

Fig. 3. Ordered assignment probability matrix for a cell size of 200 m.

best subcarriers (which does not necessarily relate to the specific terminal in any particular down-link phase but in a statistical sense it does). Next, least importance is given to the "strongest" terminals (which are closest to the base station having the lowest path loss). These terminals receive the statistically "worst" subcarriers. Furthermore, in between the strongest and the weakest terminals the assignment does not really seem to matter. There is no particular assignment preference from our results. Note that this is in accordance to the chosen optimization objective of maximizing the minimum throughput per terminal where naturally the "weakest" terminals – the ones with the lowest average SNR per subcarrier – dominate the performance metric. As the terminals closer to the base station have a much higher average SNR their specific assignment matters less for the objective function.

Next, we consider again the assignment probability matrix for the same setting as above except for choosing a larger cell size. We pick a rather drastic setting and set the radius equal to 200 m (in contrast to 100 m as in the previous example). As the cell size increases (keeping the number of terminals fixed and spacing them equidistantly), one can expect the optimization problem to become harder. The reason is that now much more terminals are critical to the objective function as their average SNR per subcarrier is rather low. This increases the dependency of the objective function on each individual chosen terminal/subcarrier pair. Even small differences in the chosen assignments can make a rather large difference regarding the resulting minimum rate per terminal. A corresponding observation for this point is that calculating the optimum for instances with a cell radius of 200 m takes much longer, indicating that the IP solver now has to do more exhaustive search operations than for the smaller cell size. In Figure 3 the resulting ordered assignment probability matrix is plotted. Notice the strong difference of this figure compared to Figure 2. While for a cell radius of 100 m the weakest terminals received the statistically "best" subcarriers (while the strongest terminals received the statistically "worst" subcarriers), the assignment probability structure changes for a radius of 200 m. In fact, only very little preference over the statistical order of subcarriers is given to terminals (for example, the strongest terminal has still a tendency to obtain statistically worse subcarriers). Instead, the weakest terminals obtain mainly a large number of subcarriers with no observed bias for them to be statistically rather good or bad. Still, the choice of the particular subcarrier is important: It is well known that especially for larger cell sizes channel-dependent multi-user OFDM resource allocation achieves a large performance gain compared to static schemes [7]. However, the way we analyze the assignment data, we do not see a statistical preference any more, possibly because the fading for the weakest terminals does not cause a subcarrier to be in a statistically overall good state (which could be fixed by normalizing $b_{j,n}^{(i)}$ to the average of each terminal - in a way similar to the proportional fair valuation criterion for subcarrier scheduling).

IV. Assignment Strategy

From the statistical observations, we develop in this section simple assignment rules to be applied to new instances of the bit matrix $\mathbf{B}^{(i)}$. The underlying intuition is straightforward: If there exist statistical preferences among the subcarrier/terminal assignments, we try to exploit them. Note that a related approach has been proposed in the context of bit-loading (for point-to-point DSL connections) in [8]. In detail, we propose to sample the ordered assignment matrix $\mathfrak{P}_I$, i.e. we fix those assignments with the highest probability with respect to their row and column (note that still for each column – i.e. for each subcarrier – only one terminal can be assigned). Denote this sampled and still ordered assignment matrix by $\mathfrak{X}_{I,\mathrm{stat}}$. Clearly, the sampled assignment matrix $\mathfrak{X}_{I,\mathrm{stat}}$ depends on the bit matrix base I considered in the statistical analysis yielding $\mathfrak{P}_I$. Given a suitable choice of $\mathfrak{X}_{I,\mathrm{stat}}$, we propose furthermore to apply this ordered assignment matrix to new instances of the bit matrix $\mathbf{B}^{(i)}$. For any such further instance, we propose to determine the set of the column and row weights – $\mathbf{c^{(i)}}$ and $\mathbf{r^{(i)}}$ – (requiring $J \cdot N$ additions), afterwards to sort them (which is both of complexity $N \log N$), and rearrange $\mathfrak{X}_{\text{stat}}$ according to the inverse reindexing of columns and rows that would be required to turn $\mathbf{B}^{(i)}$ into $\mathfrak{B}^{(i)}$. Effectively, we propose to adjust *one single, fixed and offline determined* $\mathfrak{X}_{I,\text{stat}}$ to each $\mathbf{B}^{(i)}$ simply by performing an reordering of $\mathfrak{X}_{I,\text{stat}}$ given by the column and row weights of $\mathbf{B}^{(i)}$. The computational complexity of this procedure is very low: $\mathrm{O}\left(N \log N\right)$.

The remaining issue is how to determine from a given ordered assignment matrix $\mathfrak{P}_I$ a suitable ordered assignment matrix $\mathfrak{X}_{I,\text{stat}}$. This is done by the following procedure. Along the formation of $\mathfrak{P}_I$, we form an ordered, normalized bit matrix $\mathfrak{B}_I$. Based on this matrix, we recalculate optimization problem 1 but use as input $\mathfrak{B}_I$ instead of any instance $\mathbf{B}^{(i)}$. This yields assignments which are very much in accordance to the probability regions indicated by $\mathfrak{P}_I$.

In order to judge the efficiency of our approach, we have evaluated it in comparison to the IP optimum and a static TDM approach. Considering a different set of matrices $\mathbf{B}^{(i)}$ than have been used to generate $\mathfrak{X}_{I,\text{stat}}$ we find a significant performance improvement compared to the static approach while the IP optimum clearly outperforms our statistical approach. For example, in Figure 4 we show the performance comparison for varying the delay spread of the propagation scenario between 0.1 and 0.2 μsec.. All other parameters are set according to the ones mentioned in Section III. Next, we consider different

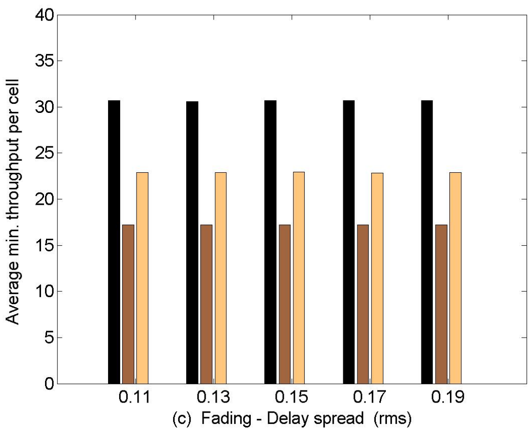

Fig. 4. Minimum average throughput per cell obtained by (a) the offline computed IP optimum (results shown in solid black) (b) the static approach (results shown in dark gray/brown) (c) the statistical approach proposed in this paper (results shown in light gray/yellow). The graph refers to the scenario described in Section III while the rms delay spread is varied.

settings of the path loss exponent α, ranging from $\alpha = 2$ to $\alpha = 3$. Note that a higher path loss exponent increases the SNR spread between strongest and weakest terminals in the cell, which was shown in Section III to lead to a less significant ordered assignment probability matrix (demonstrated for a larger cell size of 200 m). The corresponding result is shown in Figure 5. The figure with the varying path loss shows how the performance behaves of the approach as the average SNR of the strongest and weakest terminals differs either more or less compared to the case above. In either direction our

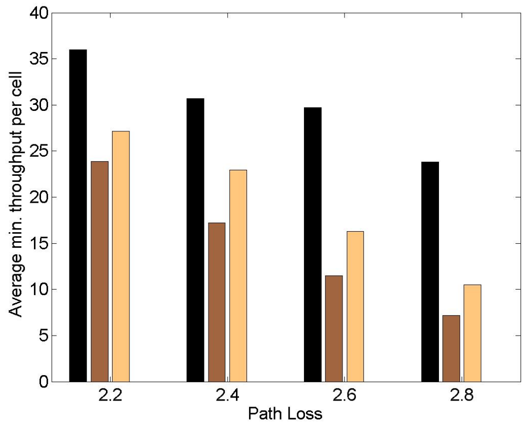

Fig. 5. Minimum average throughput per cell obtained by (a) the offline computed IP optimum (results shown in solid black) (b) the static approach (results shown in dark gray/brown) (c) the statistical approach proposed in this paper (results shown in light gray/yellow). The graph refers to the scenario described in Section III while the path loss exponent α is varied.

approach still outperforms the static approach. However, the relative difference to the IP optimum and the static approach changes - especially for a larger path loss exponent. In that case, the statistical approach as suggested achieves a much lower performance than the IP optimum (notice that for a larger path loss exponent all approaches should achieve a lower performance, however, the amount of reduction is not the same for the different approaches). This rather strong performance loss is directly related to the less significant ordered assignment probability matrix shown in Figure 3. We conclude that a different ordering criteria, possibly including a normalization to the average bit value per subcarrier per terminal, would yield a better performance.

V. Different Optimization Objectives

Above we have investigated the so called rate adaptive optimization approach, formally described by equation set 1. In addition to that, we have also investigated the statistical properties of different optimization objectives. In this section, we discuss one further optimization problem where the sum-rate of the down-link is maximized (instead of the minimum rate), constrained by a fixed amount of subcarriers each terminal should receive. Formally, this problem is given by:

$$
\begin{aligned}
\max \quad & \sum_{\forall j,n} b_{j,n}^{(i)} \cdot x_{j,n}^{(i)} \\
\text{s.t.} \quad & \sum_{\forall j} x_{j,n}^{(i)} \leq 1 \quad \forall n \\
& \sum_{\forall n} x_{j,n}^{(i)} \leq a_j \epsilon \quad \forall j
\end{aligned}
\tag{2}
$$

In this equation set a_j represents the total amount of subcarriers each terminal should receive. This is determined prior to generating the specific subcarrier/terminal assignments. For simplicity we assume in the following that a_j is fixed for all down-link phases and equals $a_j = N/J$ (rounded to the lower integer if the division yields fractions). Note that this problem is not NP hard, instead it has a graph-theoretical equivalent called bipartite weighted matching, which can be solved in polynomial time. Still, we are interested in statistical properties of the optimal assignments as well as in the performance of the ordered assignment matrix applied to instances of the bit matrix. In Figure 6 we show first the ordered assignment

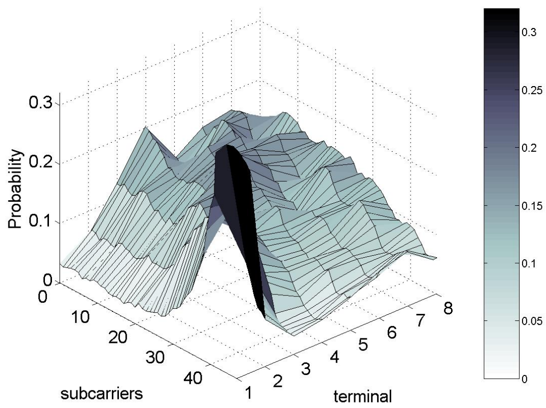

Fig. 6. Ordered assignment probability matrix for a cell size of 100 m and the objective of maximizing the sum-rate (under the constraint of assigning each terminal a fixed amount of subcarriers).

probability matrix (again considering the scenario specified in Section III) for a cell radius of 100 m. Note that in this case some parts of the terminal/subcarrier assignments have a significant structure, other parts (especially for the weaker terminals) do not have such a clear structure. The corresponding performance is shown in Figure 7 where for a varying delay spread our approach again achieves a some performance improvement. However, notice that the difference between the static and optimal approach is smaller, and hence, the achievable gain by our approach is smaller, too. Still, the proposed method can be applied to different optimization objectives yielding some improvement while the computational complexity remains the same.

VI. Conclusion

This paper discusses extremely "'light-weight"' assignment rules for channel-dependent multi-user OFDM systems. We base our rules on statistical analysis of optimal subcarrier/terminal assignments. Direct analysis of this data does not reveal any particular structure to be exploited, however, transforming the optimal assignment matrices before analyzing them according to the sum weights of columns and rows of the related instance yields interesting statistical properties, which can be exploited for simple assignment rules. In the chosen example for deriving the rule we find a significant performance advantage comparing our approach with a static one (about 30 % performance improvement). However, the approach is sensible to the SNR spread between strongest and weakest terminals in a cell. As the cell size increases (or the path loss exponent inreases) the performance gain compared to

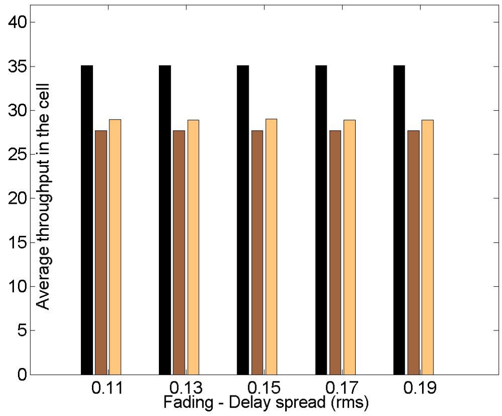

Fig. 7. Average throughput per terminal obtained by (a) the offline computed IP optimum (results shown in solid black) (b) the static approach (results shown in dark gray/brown) (c) the statistical approach proposed in this paper (results shown in light gray/yellow). The graph refers to the scenario described in Section III while the rms delay spread is varied and the sum-rate maximization is considered.

the static approach decreases (while the overall performance gain between the IP optimal assignments and a static one increases). We conclude that more sophisticated ways of statistically analyzing the optimal assignments might yield a larger performance gain. In fact, this statistical approach can also be applied to different optimization problems in the context of channel-dependent multi-user OFDM systems.

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References

[1] M. Bohge, J. Gross, M. Meyer, and A. Wolisz, "Dynamic Resource Allocation in OFDM Systems: An Overview of Cross-Layer Optimization Principles and Techniques," *IEEE Network*, vol. 21, no. 1, pp. 53–59, 2007.

[2] C. Wong, R. Cheng, K. Letaief, and R. Murch, "Multiuser OFDM with adaptive subcarrier, bit and power allocation," *IEEE Journal on Selected Areas in Communications*, vol. 17, no. 10, pp. 1747–1758, 1999.

[3] M. Ergen, S. Coleri, and P. Varaiya, "Qos aware adaptive resource allocation techniques for fair scheduling in ofdma based broadband wireless access systems," *IEEE Transactions on Broadcasting*, vol. 49, no. 4, pp. 362–370, 2003.

[4] J. Cai, X. Shen, and J. Mark, "Downlink resource management for packet transmission in OFDM wireless communication systems," *IEEE Transactions on Wireless Communications*, vol. 4, no. 4, pp. 1688–1703, 2005.

[5] J. Gross and M. Bohge, "Dynamic mechanisms in ofdm wireless systems: A survey on mathematical and system engineering contributions," Telecommunication Networks Group, Technische Universität Berlin, Tech. Rep. TKN-06-001, May 2006.

[6] S. ILOG, *ILOG CPLEX 9.0 - User's Manual*, Paris, France, 2004.

[7] M. Bohge, J. Gross, and A. Wolisz, "The Potential of Dynamic Power and Sub-carrier Assignments in Multi-User OFDM-FDMA Cells," in *Proc. of IEEE Globecom*, November 2005.

[8] L. Gao, Z. Luo, B. Tang, Y. Liu, and J. Gao, "A low-complexity adaptive bit and power allocation algorithm for MIMO-OFDM systems," in *Proc. IEEE Int. Symp. on Communications and Information Technology (ISCIT)*, 2005.